Microeconomics: Auctions

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Abstract

We first characterize the PBE in a simple first price and second price sealed bid auction with *private* values. The key result is that the expected revenue from both auctions is the same. We then look at a simple case of an auction with imperfectly known *common* value. The key result here is that a *winner's curse* arises in some states of the world.

1 Auctions with private and independent values: The revenue equivalence theorem

Auctions with private and independent values can take several forms (see BD p267):

- English auction
- Dutch auction
- First-price sealed bid auction
- Second-price sealed bid auction (aka Vickrey auction)

They all yield the same expect revenue to the seller. We illustrate this point with an exemple.

1.1 Exemple

- One seller. The set of buyers is $i = 1, ..., N + 1$.
- Buyers' private valuation for an object is θ_i and are iid $\theta_i \tilde{} U[0, 1]$.
- Their action is a bid, with $b_i(\theta_i) \geq 0$, all i.

- Buyer i's expected payoff is

$$\mathrm{E}\pi_i\left(\theta_i; \{b_j\}_{j=1}^{N+1}\right) = \Pr\{b_i = \max_j b_j\}\left(\theta_i - B_i\right),$$

  where $B_i = b_i$ in the first price sealed bid auction and $B_i = \max_{j\neq i} b_j$ in the Vickrey auction. See below for the seller's expected payoff.

1.2 First-price sealed bid auction

Let us guess the form of a symmetric PBE equilibrium. With a uniform distribution, assume

$$b_j\left(\theta_j\right) = \beta\theta_j,$$

some $\beta > 0$ to be determined at equilibrium. Assuming that

$$b_i\left(\theta_i\right) = \beta_i\theta_i,$$

we want to ensure that $\beta_i = \beta$ is a best reply for i.[1]

Under the biding strategies laid out above,

$$\begin{aligned}\Pr\{b_i &= \max_j b_j\} = \prod_{j\neq i}\Pr\{b_j < b_i\} \\ &= \Pr\left\{\theta_j < \frac{\beta_i}{\beta}\theta_i\right\}^N \\ &= \left(\frac{\beta_i}{\beta}\theta_i\right)^N.\end{aligned}$$

Thus

$$\max_{\beta_i}\mathrm{E}\pi_i\left(\theta_i; \{b_j\}_{j=1}^{N+1}\right) = \max_{\beta_i}\left(\frac{\beta_i}{\beta}\theta_i\right)^N\left(1-\beta_i\right)\theta_i$$

yields

$$0 = \left(\frac{\beta_i}{\beta}\theta_i\right)^{N-1}\frac{\theta_i}{\beta}\left[N\left(1-\beta_i\right)-\beta_i\right]$$

so that, for all i,

$$\beta_i = \frac{N}{N+1} \equiv \beta.$$

[1] This is the unique symmetric BNE with strictly increasing and differentiable strategy functions; see Gibbons Appendix 3.2.B (p157).

Using $G(\theta) \equiv \Pr\{\theta_{\max} < \theta\} = \prod_{i=1}^{N+1} \Pr\{\theta_i < \theta\} = \theta^{N+1}$, the expected revenue for the seller is

$$\begin{aligned}
\mathrm{E}(\pi_{seller}) &= \int_0^1 \beta\theta \mathrm{d}G(\theta) \\
&= \int_0^1 \beta\theta (N+1)\theta^N \mathrm{d}\theta \\
&= \beta\frac{N+1}{N+2} \\
&= \frac{N}{N+2}.
\end{aligned}$$

1.3 Second-price sealed bid auction (aka Vickrey auction)

Here it is a weakly dominant strategy to bid

$$b_i = \theta_i,$$

for all i.

The expected revenue for the seller is

$$\begin{aligned}
\mathrm{E}(\pi_{seller}) &= \int_0^1 \int_0^{\theta_{\max}} \theta \mathrm{d}G(\theta|\, \theta < \theta_{\max}) \mathrm{d}G(\theta_{\max}) \\
&= \int_0^1 \left[\int_0^{\theta_{\max}} \theta N \left(\frac{\theta}{\theta_{\max}}\right)^N \frac{1}{\theta}\mathrm{d}\theta\right](N+1)\theta_{\max}^N \mathrm{d}\theta_{\max} \\
&= N(N+1)\int_0^1 \frac{1}{\theta_{\max}^N}\left[\frac{\theta^{N+1}}{N+1}\right]_0^{\theta_{\max}} \theta_{\max}^N \mathrm{d}\theta_{\max} \\
&= \frac{N}{N+2}\left[\theta_{\max}^{N+2}\right]_0^1 \\
&= \frac{N}{N+2},
\end{aligned}$$

which is the same as above, as was to be shown.

2 Auctions with imperfectly known common values: The winner's curse

Winning the auction brings mixed news: the loosers may have got a very pessimistic signal. We illustrate this point with a simple exemple.

2.1 Exemple

- Players: 2 buyers and one seller.
- The value ν of the object can take two values with equal probability, $\nu \in \{H, L\}$, with $H > L > 0$.
- Each buyer gets a signal $s(i) \in \{s_H, s_L\}$, with

$$\Pr\{H | s_H\} = \Pr\{L | s_L\} \equiv p > \frac{1}{2}$$

so that

$$\nu_H \equiv \mathrm{E}[\nu | s_H] = pH + (1-p)L = L + p(H-L)$$

and

$$\nu_L \equiv \mathrm{E}[\nu | s_L] = pL + (1-p)H = L + (1-p)(H-L).$$

- Strategies are contingent on the signal so that we may write $b_i = b(s(i))$.
- In order to compute expected payoffs, we may write

$$\begin{aligned} \mathrm{E}[\nu | s_L, s_L] - b(s_L) &= \frac{p^2 L + (1-p)^2 H}{p^2 + (1-p)^2} - b(s_L) \\ &= [\nu_L - b(s_L)] - \frac{p^2(1-p)}{p^2 + (1-p)^2}(H-L), \end{aligned}$$

so that

$$b(s_L) < \nu_L,$$

for otherwise the bidder who got a negative signal would make a loss. Put differently, if I win the auction having received a low signal then this implies that the other buyer also got a low signal, so that the state of nature is of type L with a probability higher than the ex-ante probability p; hence the '*curse*.'

- Also,

$$\begin{aligned} \mathrm{E}[\nu | s_H, s_H] - b(s_H) &= \frac{p^2 H + (1-p)^2 L}{p^2 + (1-p)^2} - b(s_H) \\ &= [\nu_H - b(s_H)] + p\frac{(2p-1)(1-p)}{p^2 + (1-p)^2}(H-L), \end{aligned}$$

so that $E[\nu | s_H, s_H] - b(s_H) > 0$ if

$$b(s_H) = \nu_H,$$

that is, if both players receive a positive signal (and knew about it) then they would revise their beliefs upwards – this is because the ex-post probability of a high state of nature is more than p, a '*blessing*.'

- Finally,

$$\mathrm{E}\left[\nu|\, s_L, s_H\right] - b\left(s_H\right) = 0$$

(under the reasonable condition, verified at equilibrium, that $b\left(s_H\right) > b\left(s_L\right)$, so that the player receiving a high signal wins the auction) and

$$\begin{aligned}\mathrm{E}\left[\nu|\, s_H, s_L\right] - b\left(s_H\right) &= \frac{H+L}{2} - b\left(s_H\right) \\ &= \left[\nu_H - b\left(s_H\right)\right] + \left(p - \frac{1}{2}\right)(H-L)\end{aligned}$$

so that $E\left[\nu|\, s_H, s_L\right] - b\left(s_H\right) > 0$ if $b\left(s_H\right) = \nu_H$,that is, if the player receiving a positive signal would revise her beliefs upwards.

- Thus, in a Wickley auction, the PBE equilibrium bids are

$$b\left(s_L\right) = \mathrm{E}\left[\nu|\, s_L, s_L\right]$$

and

$$b\left(s_H\right) = \mathrm{E}\left[\nu|\, s_H, s_H\right],$$

which implies $E\left[\nu|\, s_H, s_L\right] - b\left(s_H\right) = \frac{1}{2}\frac{(2p-1)^3}{p^2+(1-p)^2} > 0$, and

$$\mathrm{E}\left(\pi_{seller}\right) = \left(p - \frac{1}{2}\right)(H-L) + \frac{p^2 L + (1-p)^2 H}{p^2 + (1-p)^2}.$$