Agglomeration Economies: Matching and Sharing

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Abstract

Cities are more productive as a result of a better *matching* between workers' skills and firms' skill requirements in the model due to Sato and Thisse (2007, European Economic Review); as a result of a finer division of labour in the model due to Becker and Murphy (1992, Quarterly Journal of Economics); and as a result of input sharing in the Ethier (1982, American Economic Review) version of the Dixit-Stiglitz model.

1 Matching: The Sato-Thisse model

The set up is as follows:

- Final good freely traded, constant returns to scale, perfect competition: $p = 1$.
- Endogenous number of firms: n.
- Exogenous city labour force: L.
- Heterogenous skills distributed uniformly around unit circle; s skills type of arbitrary worker; s_k skill requirement of firm k. s is private information.
- Training costs borne by worker s if works at firm s_k: $\tau |s_k - s|$, some $\tau > 0$.
- Firm s_k pays same wage w_k to all its workers. Local monopsony power.
- IRS at firm level: $C(x_k; r, w_k) = \alpha r + \beta w_k x_k$, where r is the economy-wide price of capital and x_k is output.

1.1 Equilibrium

Let $\overline{s_k}$ denote the worker indifferent between working for firm s_k and firm s_{k+1} and let $\underline{s_k}$ denote the worker indifferent between working for firm s_k and firm s_{k-1}. Then, by definition,

$$\begin{aligned} \underline{s_k} &= \frac{w_{k-1} - w_k + \tau\left(s_{k-1} + s_k\right)}{2\tau} \\ \overline{s_k} &= \frac{w_k - w_{k+1} + \tau\left(s_{k+1} + s_k\right)}{2\tau}. \end{aligned}$$

Profits are thus

$$\pi_k = \frac{L\left(\overline{s_k} - \underline{s_k}\right)}{\beta}\left(1 - \beta w_k\right) - \alpha r.$$

Each firm chooses w_k non-cooperatively (taking all other firms' wage as given) so as to maximise profit. The first-oder condition to this program is (droping k subscript):

$$0 = \frac{\partial \pi}{\partial w} \equiv \frac{L}{\beta}\left[-\beta\left(\overline{s_k} - \underline{s_k}\right) + \frac{1}{\tau}\left(1 - \beta w\right)\right].$$

It is easy to show that the second-order condition holds. At a symmetric equilibrium, all firms hire the same number of workers so that $\overline{s_k} - \underline{s_k} = 1/n$. This implies

$$w^{NE} = \frac{1}{\beta} - \frac{\tau}{n},$$

which is increasing in n by inspection.

Free-entry:

$$0 = \pi^{NE} \equiv \frac{L\tau}{n^2} - \alpha r.$$

This implies

$$n^* = \sqrt{\frac{L\tau}{\alpha r}}$$

and

$$w^* = \frac{1}{\beta} - \sqrt{\frac{\tau\alpha r}{L}}.$$

The maximum distance between a firm's skill requirement and the worker that it hires is $1/(2n)$ so that the average distance is $1/(4n)$. As a result, the *average wage net of training costs* ('net wages' henceforth) is equal to $w^* - \tau/\left(4n^*\right)$, i.e.

$$\omega^* = \frac{1}{\beta} - \frac{5}{4}\sqrt{\frac{\tau\alpha r}{L}} < \frac{1}{\beta}.$$

The inequality is the result of firms' monopsony power.

Full-employment of labour requires

$$n\beta x = L$$

so that gross output is (trivially)

$$Y^* \equiv n^* x^* = \frac{L}{\beta}.$$

Per capita income is thus independant of city size. This is because (i) training costs are borne in 'effort', not lost productivity, and (ii) fixed costs are borne in (country-wide) capital, not (local) labour. I relax the latter assumption below. Relaxing either yields again $\partial(Y^*/L)/\partial L > 0$.

1.2 Agglomeration economies

Nominal and average net wages are increasing in city size:

$$\begin{aligned} w^* &= \frac{1}{\beta} - \sqrt{\frac{\tau\alpha r}{L}} \\ \overline{\omega}^* &= w^* - \tau\frac{1}{4n^*} = \frac{1}{\beta} - \frac{5}{4}\sqrt{\frac{\tau\alpha r}{L}}. \end{aligned}$$

Nominal wages are increasing in city size because size reduces monopsony power. Net wages are increasing in city size because size reduces monopsony power *and* training costs are reduced by improved matching.

Note: the *variance* of net wages is decreasing in city size:

$$\mathrm{Var}\,(\omega) = \frac{1}{3}\left(\frac{\tau}{4n^*}\right)^2 = \frac{\alpha r \tau}{48L}.$$

The model has nothing to say about the variance of nominal wages (constant here). Empirically, city wage inequality is increasing in city size.

1.3 Aside: The Sato-Thisse model with one factor of production[1]

Now the cost function is $C\left(x_k, w_k\right) = \left(\alpha + \beta x_k\right) w_k$ and profits are

$$\pi_k = \frac{L\left(\overline{s_k} - \underline{s_k}\right)}{\beta}\left(1 - \beta w_k\right) - \alpha w_k.$$

At the symmetric Nash equilibrium wages and profits are equal to

$$\begin{aligned} w^{NE} &= \frac{1}{\beta} - \frac{\tau}{n} - \frac{\alpha\tau}{L} \\ \pi^{NE} &= -\frac{\alpha}{\beta} + \frac{\alpha^2\tau}{L} + \frac{2\alpha\tau}{n} + \frac{L\tau}{n^2}, \end{aligned}$$

which is increasing in L only if $n < L/\alpha$. Thus, the relevant root of the free-entry condition is

$$n^* = \frac{L}{\alpha}\frac{\alpha\beta\tau + \sqrt{\alpha\beta\tau L}}{L - \alpha\beta\tau}$$

[1] Details in SatoThisse.mw.

and the minimum size a city must reach to sustain production is $L_{\min} \equiv (2\alpha)^2 \beta\tau$.

Per capita production (nominal wage) is equal to

$$y^* = w^* = \frac{1}{\beta}\left(1 - \frac{\alpha\beta\tau + \sqrt{\alpha\beta\tau L}}{L - \alpha\beta\tau}\right),$$

which is smaller than labour productivity $1/\beta$ by the producers' monopsony power and increasing in L by

$$\frac{\partial y^*}{\partial L} = \frac{1}{2}\sqrt{\frac{\alpha\tau}{\beta L}}\left(\frac{\sqrt{L} + \sqrt{\alpha\beta\tau}}{L - \alpha\beta\tau}\right)^2 > 0.$$

By the same token,

$$\omega^* \equiv w^* - \frac{\tau}{4n^*} = \frac{1}{\beta} - \frac{5\tau}{4n^*} - \frac{\alpha\tau}{L}$$

is also increasing in L.

2 Sharing: The Becker-Murphy model

The set up is as follows:

- Final good freely traded, perfect competition: $p = 1$.
- Production of final good requires continuum of tasks over the unit interval; $s \in [0, 1]$.
- CES technology: $Y = \left[\int_0^1 Y(s)^{\frac{1}{1+\varepsilon}} ds\right]^{1+\varepsilon}$, some $\varepsilon > 0$.
- Exogenous team size: L, so that each worker operates $1/L$ tasks.
- Workers need time to *learn* skills; $t(s)$ denotes the amount of time devoted to learn skill s.
- Productivity in skill s is increasing in time spent learning s: $E(s) = t(s)^\theta$, some $\theta \in (0, 1)$.
- Time is a scarce ressource: each worker is endowed with 1 unit of time. If she devotes the same amount of time to each task (which will be the case at equilibrium), then the amount of time spent operating taks s is $T(s) = L - t(s)$.
- Production of tasks s is thus equal to $Y(s) = [L - t(s)]\, t(s)^\theta$.

2.1 Equilibrium

Each task is performed by only one worker; each worker specializes in $1/L$ tasks. Workers maximize their total production:

$$\max_t x \equiv \frac{1}{L}(L - t)\, t^\theta.$$

First-order condition:

$$0 = \frac{\partial x}{\partial t} \equiv \theta t^{-1+\theta} - \frac{1+\theta}{L} t^{\theta}.$$

It is easy to show that the second-order condition holds.

Solving the first-order condition yields that workers devote the amount of time

$$t^* = \frac{\theta}{1+\theta} L$$

to learning each skill she specialises in.

Aggregate team production is

$$Y^* = \frac{\theta^{\theta}}{(1+\theta)^{1+\theta}} L^{1+\theta}.$$

2.2 Agglomeration economies

Output per capita is increasing in city size, L, by inspection:

$$\frac{Y^*}{L} = \frac{\theta^{\theta}}{(1+\theta)^{1+\theta}} L^{\theta}.$$

From

$$t^* = \frac{\theta}{1+\theta} L$$

it is clear that workers spend more time learning each skill if they belong to a large team than if they belong to a small team (i.e. $\partial t^* / \partial L > 0$). Overall, each worker devotes a constant fraction

$$\frac{t^*}{L} = \frac{\theta}{1+\theta}$$

of her time learning and the complementary fraction

$$1 - \frac{t^*}{L} = \frac{1}{1+\theta}$$

producing *per task*.[2] This is not a general result. Rather, it is the net effect of two opposing forces: each worker specialises in fewer tasks but learn more of each.

2.3 Important remark

Agglomeration economies are the result of increased sharing at the *intensive margin* in the Becker-Murphy model. By contrast, agglomeration economies are the result of increased sharing at the *extensive margin* in the Dixit-Stiglitz-Ethier model. We turn to this model next.

[2]Together with the fact that each worker belonging to a large team focuses on a narrow set of tasks, this implies that specialisation enables workers to devote more time producing. This may look at odds with the fact that cities are good places in which to learn.

3 Sharing: The Dixit-Stiglitz-Ethier model

The set up is as follows:

- Endogenous number of input producers: n. Intermediate producers are monopolistically competitive.
- Exogenous city labour force: L.
- Final good freely traded, constant returns to scale, perfect competition: $p = 1$.
- Production of final good requires continuum of horizontally differentiated inputs.
- CES technology: $Y = \left[\int_0^n x(s)^{\frac{1}{1+\varepsilon}} ds\right]^{1+\varepsilon}$, some $\varepsilon > 0$.
- IRS at the firm level: $C(x) = (\alpha + \beta x) w$.

3.1 Equilibrium

Final good firms minimize cost, $c \equiv \int_0^n p(s) x(s) \mathrm{d}s$ subject to attaining output level Y:

$$\min_{x(s)} \int_0^n p(s) x(s) \, \mathrm{d}s + \lambda \left\{ Y - \left[\int_0^n x(s)^{\frac{1}{1+\varepsilon}} ds \right]^{1+\varepsilon} \right\},$$

where λ is the lagrangian multiplier. Recall that its economic interpretation is the marginal cost.

Focs:

$$0 = \frac{\partial c}{\partial x(s)} \equiv p(s) - \lambda Y^{\frac{\varepsilon}{1+\varepsilon}} x(s)^{\frac{1}{1+\varepsilon} - 1}.$$

Rearranging and integrating yields

$$\lambda = \left[\int_0^n p(s)^{-1/\varepsilon} ds \right]^{-\varepsilon} \equiv P.$$

In words: the marginal cost of the final good sector is equal to the price index of intermediates.

Monopolistically competitive intermediate producers set optimal prices

$$p(s) = p \equiv (1 + \varepsilon) \beta w,$$

all s.

Free-entry drives their profits to zero:

$$0 = \pi(s) \equiv [\varepsilon \beta x(s) - \alpha] w$$

so that $x(s) = x \equiv \alpha / (\varepsilon \beta)$ for all s. Note in particular that x is independant of city size L.

Full-employment of labour requires

$$L = n\left(\alpha + \beta x\right) = n\alpha\frac{1+\varepsilon}{\varepsilon}$$

so that the equilibrium number of firms is proportional to city size:

$$n^{*} = L\frac{\varepsilon}{\alpha\left(1+\varepsilon\right)}.$$

Free-trade and perfect competition in final goods yields $\lambda = 1$ which implies

$$\begin{aligned} 1 &= \left[\int_0^n p\left(s\right)^{-1/\varepsilon} ds\right]^{-\varepsilon} \\ &= n^{-\varepsilon}\left(1+\varepsilon\right)\beta w \end{aligned}$$

so that equilibrium nominal and net wages are

$$w^{*} = \frac{1}{\left(1+\varepsilon\right)^{1+\varepsilon}\beta}\left(\frac{\varepsilon L}{\alpha}\right)^{\varepsilon}.$$

Finally, aggregate production is equal to $Y = n^{1+\varepsilon}x$ so that, at equilibrium,

$$Y^{*} = \frac{1}{\beta\left(1+\varepsilon\right)^{1+\varepsilon}}\left(\frac{\varepsilon}{\alpha}\right)^{\varepsilon}L^{1+\varepsilon}.$$

3.2 Agglomeration economies

Output per capita is increasing in city size, L, by inspection:

$$\frac{Y^{*}}{L} = \frac{1}{\beta\left(1+\varepsilon\right)^{1+\varepsilon}}\left(\frac{\varepsilon L}{\alpha}\right)^{\varepsilon}.$$

Worker productivity (as measured by their wage w^{*}) is also increasing in city size: indeed, $w^{*} = Y^{*}/L$ because labour is the sole factor of production in this model.

Ouput per capita and worker productivity are increasing in city size because larger cities enable final good producers to share a larger pool of intermediate producers – an *extensive margin*.